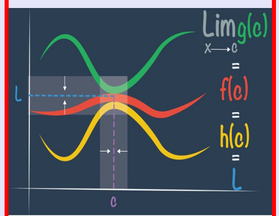
Calculus I

Lecture 16



Feb 19-8:47 AM

Introduction To integration
(Reverse of Derivatives)

$$\int f'(x)\,dx = f(x) + C$$

Integral Integrand with respect to Indefinite integral

$$\frac{d}{dx}\left[f(x) + C\right] = f'(x)$$

$$\int 5\,dx = 5x + C$$

$$\int -2\,dx = -2x + C$$

$$\boxed{\int k\,dx = kx + C}$$

$$\int 2x\,dx = x^2 + C$$

$$\boxed{\int k f(x)\,dx = k\int f(x)\,dx}$$

$$\int 5x\,dx = \int \frac{5}{2}\cdot 2x\,dx = \frac{5}{2}\int 2x\,dx$$

$$= \frac{5}{2}x^2 + C$$

Feb 2-8:06 AM

$$\int (2x+5)\,dx = x^2 + 5x + C$$

$$\int (3x^2 - 2x + 8)\,dx = x^3 - x^2 + 8x + C$$

$$\boxed{\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx}$$

$$\int (5x^4 - 4x^3 + \cos x)\,dx = x^5 - x^4 + \sin x + C$$

Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

Feb 2-8:15 AM

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \boxed{\frac{2}{3}x^{\frac{3}{2}} + C}$$

$$x^{\frac{3}{2}} = (\sqrt{x})^3 = (\sqrt{x})^2\sqrt{x}$$

$$= x\sqrt{x}$$

$$= \boxed{\frac{2}{3}x\sqrt{x} + C}$$

$$\int x\sqrt{x}\,dx = \int x^{\frac{3}{2}}\,dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C$$

$$= \frac{x^{5/2}}{5/2} + C$$

$$= \frac{2}{5}x^{5/2} + C$$

$$= \boxed{\frac{2}{5}x^2\sqrt{x} + C}$$

Feb 2-8:23 AM

Find $\int (x^6 + \sec^2 x - \sqrt[3]{x})dx$

$$= \int x^6 dx + \int \sec^2 x \, dx - \int x^{1/3} dx$$

$$= \frac{x^7}{7} + \tan x - \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C$$

$$= \frac{1}{7}x^7 + \tan x - \frac{x^{4/3}}{4/3} + C$$

$$= \frac{1}{7}x^7 + \tan x - \frac{3}{4}x^{4/3} + C$$

$$= \boxed{\frac{1}{7}x^7 + \tan x - \frac{3}{4}x\sqrt[3]{x} + C}$$

Find $\int (4x^4 - \csc^2 x)dx$

$$= \int 4x^4 dx - \int \csc^2 x \, dx = 4\int x^4 dx + \int -\csc^2 x \, dx$$

$$= \boxed{\frac{4}{5}x^5 + \cot x + C}$$

Feb 2-8:30 AM

Integration Rules

1) $\int K f(x)dx = K\int f(x)dx$

2) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

3) $\int K dx = Kx + C$

4) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$

5) $\int \sin x \, dx = -\cos x + C$

6) $\int \cos x \, dx = \sin x + C$

7) $\int \sec^2 x \, dx = \tan x + C$

8) $\int \csc^2 x \, dx = -\cot x + C$

9) $\int \sec x \tan x \, dx = \sec x + C$

10) $\int \csc x \cot x \, dx = -\csc x + C$

Feb 2-8:38 AM

$$\int 2x(x^2-3x+5)\,dx$$

$$= \int [2x^3 - 6x^2 + 10x]\,dx$$

$$= \frac{2x^4}{4} - \frac{6x^3}{3} + \frac{10x^2}{2} + C$$

$$= \boxed{\frac{1}{2}x^4 - 2x^3 + 5x^2 + C}$$

$\frac{1}{x^2} = x^{-2}$

Find $\int \left(x^2 - 2 + \frac{1}{x^2}\right)dx$

$$= \frac{x^3}{3} - 2x + \frac{x^{-2+1}}{-2+1} + C$$

$$= \frac{1}{3}x^3 - 2x + \frac{x^{-1}}{-1} + C$$

$$= \boxed{\frac{1}{3}x^3 - 2x - \frac{1}{x} + C}$$

Feb 2-8:45 AM

Find

1) $\int \sec x(\sec x - \tan x)\,dx$

$$= \int (\sec^2 x - \sec x \tan x)\,dx = \boxed{\tan x - \sec x + C}$$

2) $\int \frac{\sin 2x}{\sin x}\,dx = \int \frac{2\cancel{\sin x}\cos x}{\cancel{\sin x}}\,dx$

$$= \int 2\cos x\,dx = \boxed{2\cdot \sin x + C}$$

Feb 2-8:54 AM

3) $\int \frac{3x - 2}{\sqrt{x}}\,dx$

$= \int \left[\frac{3x}{\sqrt{x}} - \frac{2}{\sqrt{x}}\right]dx = \int \left(3\sqrt{x} - \frac{2}{\sqrt{x}}\right)dx$

$= \int \left[3x^{1/2} - 2x^{-1/2}\right]dx$

$= 3 \cdot \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 2 \cdot \frac{x^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + C$

$= 3 \cdot \frac{x^{3/2}}{3/2} - 2 \cdot \frac{x^{1/2}}{\frac{1}{2}} + C$

$= \cancel{3} \cdot \frac{2}{\cancel{3}} x^{3/2} - 2 \cdot \frac{2x^{1/2}}{1} + C$

$= \boxed{2x\sqrt{x} - 4\sqrt{x} + C}$

Feb 2-9:00 AM

4) $\int (x^2+4)(x+2)(x-2)\,dx$

$= \int (x^4 - 16)\,dx = \boxed{\frac{x^5}{5} - 16x + C}$

5) $\int (\underbrace{1 + \tan^2 x}_{\text{Trig.}})\,dx = \int \sec^2 x\,dx$

$= \boxed{\tan x + C}$

Feb 2-9:06 AM

Definite Integral

$$\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a)$$

$f'(x)$ must be cont. on $[a,b]$

$$\int_1^3 2x\,dx = x^2\Big|_1^3 = 3^2 - 1^2 = 9 - 1 = \boxed{8}$$

$$\int_0^{\pi} \cos x\,dx = \sin x\Big|_0^{\pi} = \sin\pi - \sin 0 = 0 - 0 = \boxed{0}$$

$$\int_4^9 \sqrt{x}\,dx = \int_4^9 x^{1/2}\,dx = \frac{x^{3/2}}{3/2}\Big|_4^9 = \frac{2}{3}x\sqrt{x}\Big|_4^9$$

$$= \frac{2}{3}\left[9\sqrt{9} - 4\sqrt{4}\right]$$

$$= \frac{2}{3}\left[27 - 8\right]$$

$$= \frac{2}{3}\cdot 19 = \boxed{\frac{38}{3}}$$

Feb 2-9:25 AM

Evaluate $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big|_0^{\pi/4}$

$$= \tan\frac{\pi}{4} - \tan 0$$

$\tan\frac{\pi}{4} \to 1$, $\tan 0 \to 0$

$$= \boxed{1}$$

Evaluate $\displaystyle\int_0^2 (2x-3)(4x^2+1)\,dx$

$$= \int_0^2 \left[8x^3 + 2x - 12x^2 - 3\right]dx$$

$$= \left(\frac{8x^4}{4} + x^2 - \frac{12x^3}{3} - 3x\right)\Big|_0^2$$

$$= \left(2x^4 + x^2 - 4x^3 - 3x\right)\Big|_0^2$$

$$= \left[2(2)^4 + 2^2 - 4(2)^3 - 3(2)\right] - 0$$

$$= 32 + 4 - 32 - 6 = \boxed{-2}$$

Evaluate $\displaystyle\int_{\pi}^{\pi} \left[4\sin x - 3\cos x\right]dx = \boxed{0}$

$$\boxed{\int_a^a f(x)\,dx = 0}$$

Feb 2-9:35 AM

Evaluate $\int_1^2 (3x^2 - 2x)\,dx$

$= (x^3 - x^2)\Big|_1^2 = (2^3 - 2^2) - (1^3 - 1^2)$

$= 4 - 0$

$= \boxed{4}$

what about

$\int_2^1 (3x^2 - 2x)\,dx$? $\boxed{-4}$

$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$$

Feb 2-9:44 AM

Evaluate $\int_1^2 \left(\underbrace{\frac{1}{x^2} - \frac{4}{x^3}}_{\text{undefined at } x=0}\right)dx$

is 0 in the interval [1,2]?

No.

$\int_1^2 (x^{-2} - 4x^{-3})\,dx$

$= \left(\frac{x^{-2+1}}{-2+1} - \frac{4x^{-3+1}}{-3+1}\right)\Big|_1^2 = \left(\frac{x^{-1}}{-1} - \frac{4x^{-2}}{-2}\right)\Big|_1^2$

$= \left(-\frac{1}{x} + \frac{2}{x^2}\right)\Big|_1^2 = \left(\frac{2}{x^2} - \frac{1}{x}\right)\Big|_1^2$

$= \cancel{\left(\frac{2}{4} - \frac{1}{2}\right)}^{\,0} - \left(\frac{2}{1} - \frac{1}{1}\right) = -(2-1) = \boxed{-1}$

Feb 2-9:48 AM

Evaluate $\int_{\pi/4}^{\pi/3} \csc^2\theta \, d\theta$

$$= -\cot\theta \Big|_{\pi/4}^{\pi/3} = -\left(\cot\frac{\pi}{3} - \cot\frac{\pi}{4}\right)$$

$$= -\left(\cot 60^\circ - \cot 45^\circ\right)$$

$$= -\left(\frac{\sqrt{3}}{3} - 1\right) = \boxed{1 - \frac{\sqrt{3}}{3}}$$

Evaluate $\int_{-1}^{1} \frac{1}{x^2}\, dx.$

$\hookrightarrow$ Not defined at $x = 0$.

0 is in $[-1, 1]$

wait for Calc. II.

Feb 2-9:54 AM

Evaluate $\int_0^{\pi/3} \frac{\sin x + \sin x \tan^2 x}{\sec^2 x}\, dx$

$$= \int_0^{\pi/3} \frac{\sin x \cancel{(1 + \tan^2 x)}}{\cancel{\sec^2 x}}\, dx$$

$$= \int_0^{\pi/3} \sin x \, dx = -\cos x \Big|_0^{\pi/3}$$

$$= -\left(\cos\frac{\pi}{3} - \cos 0\right) = -\left(\frac{1}{2} - 1\right) = \boxed{\frac{1}{2}}$$

Feb 2-10:01 AM

Evaluate $\displaystyle\int_0^{\pi/4} \frac{1+\cos^2 x}{\cos^2 x}\,dx$

$$\frac{1+\cos^2 x}{\frac{1}{\sec^2 x}} = \sec^2 x\,(1+\cos^2 x) = \sec^2 x + \sec^2 x \cos^2 x = \sec^2 x + 1$$

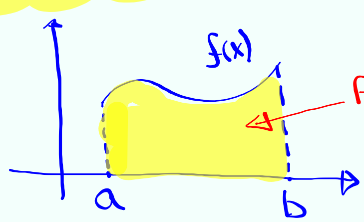
$$= \int_0^{\frac{\pi}{4}} \left[\frac{1}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}\right] dx$$

$$= \int_0^{\pi/4} \left(\sec^2 x + 1\right) dx = \left(\tan x + x\right)\Big|_0^{\pi/4}$$

$$= \left(\underbrace{\tan\tfrac{\pi}{4}}_{1} + \tfrac{\pi}{4}\right) - \left(\underbrace{\tan 0}_{0} + 0\right) = \boxed{1 + \tfrac{\pi}{4}}$$

Feb 2-10:06 AM

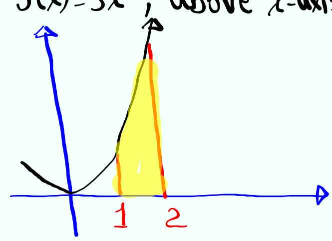
If $f(x) \geq 0$ on $[a,b]$ and $f(x)$ is cont. on $[a,b]$, then the area below $f(x)$ and above x-axis on $[a,b]$ is



$$\text{Area} = \int_a^b f(x)\,dx$$

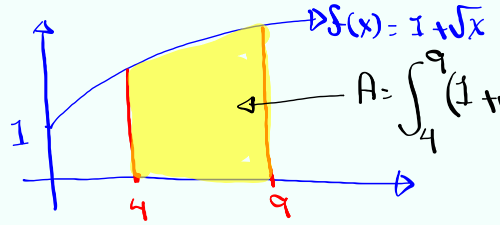
ex: Find the area below $f(x) = 3x^2$, above x-axis from $x = 1$ to $x = 2$.

$$A = \int_1^2 3x^2\,dx = x^3\Big|_1^2 = 2^3 - 1^3 = \boxed{7}$$



Feb 2-10:13 AM

Find the area below $f(x) = 1 + \sqrt{x}$, above x-axis from $x = 4$ to $x = 9$.



$x^{1/2}$

$$A = \int_4^9 (1 + \sqrt{x})\,dx$$

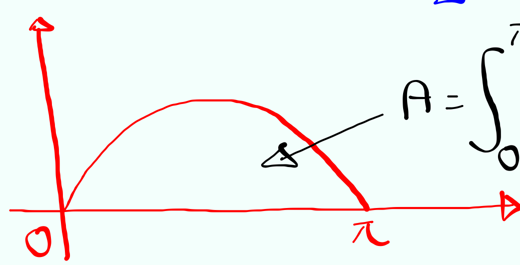
$$= \left(x + \frac{x^{3/2}}{3/2}\right)\Bigg|_4^9 = \left(x + \frac{2}{3}x\sqrt{x}\right)\Bigg|_4^9$$

$$= \left(9 + \frac{2}{\cancel{3}}\cdot \cancel{9}\sqrt{9}\right) - \left(4 + \frac{2}{3}\cdot 4\sqrt{4}\right)$$

$$= (9 + 18) - \left(4 + \frac{16}{3}\right) = 27 - \frac{28}{3} = \boxed{\frac{53}{3}}$$

Feb 2-10:19 AM

Find the area below $f(x) = \sin x$, above the x-axis on $[0, \pi]$.



$$A = \int_0^{\pi} \sin x\,dx$$

$$= -\cos x \Big|_0^{\pi}$$

$$= -[\cancel{\cos \pi}^{-1} - \cancel{\cos 0}^{1}]$$

$$= -[-1 - 1] = \boxed{2}$$

Feb 2-10:26 AM

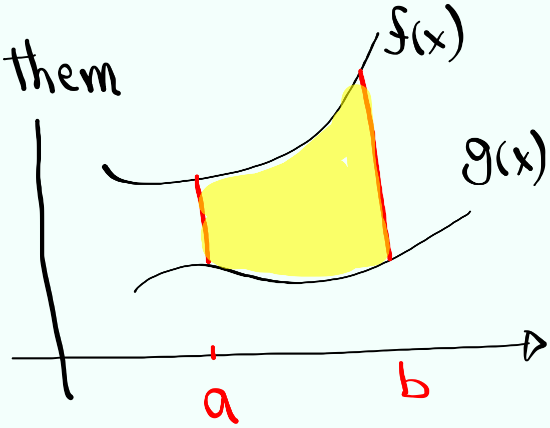
Area between two Curves:

Suppose $f(x) \geq g(x)$ on $[a,b]$

Area between them

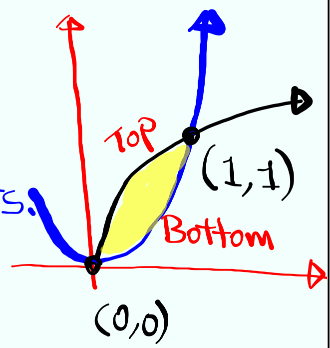
$$A = \int_a^b (\text{Top} - \text{Bottom})\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx$$



Feb 2-10:30 AM

1) Graph $f(x) = \sqrt{x}$ and $g(x) = x^2$



2) Find their intersection pts.

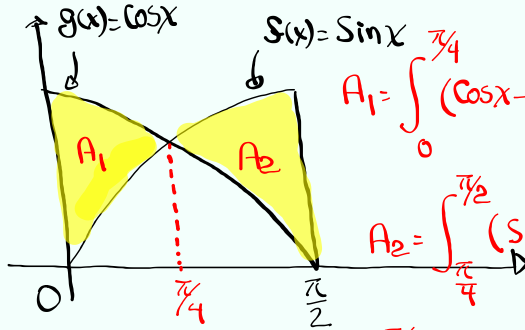
3) Find the enclosed area between them

$$A = \int_0^1 [\sqrt{x} - x^2]\,dx = \left(\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$= \left(\frac{2}{3}x\sqrt{x} - \frac{1}{3}x^3\right)\Big|_0^1 = \boxed{\frac{1}{3}}$$

Feb 2-10:33 AM

find the shaded area below



$$A_1 = \int_0^{\pi/4} (\cos x - \sin x)\,dx =$$

$$A_2 = \int_{\pi/4}^{\pi/2} (\sin x - \cos x) =$$

$$A_1 = \left[\sin x - (-\cos x)\right]\Big|_0^{\pi/4} = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin 0 - \cos 0$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1$$

$$= \boxed{\sqrt{2} - 1}$$

$$\boxed{A_2 = \sqrt{2} - 1} \checkmark$$

Total shaded Area

$2\sqrt{2} - 2$

Feb 2-10:39 AM

Evaluate $\int 2x(x^2+1)^2\,dx = \int (2x^5 + 4x^3 + 2x)\,dx$

$(x^2+1)^2 = x^4 + 2x^2 + 1$ $\quad = \frac{2x^6}{6} + \frac{4x^4}{4} + \frac{2x^2}{2} + C$

$2x(x^2+1)^2 = 2x^5 + 4x^3 + 2x$ $\quad = \boxed{\frac{1}{3}x^6 + x^4 + x^2 + C}$

Let $u = x^2 + 1$ $\quad du = 2x\,dx$

$$\int 2x(x^2+1)^2\,dx = \int u^2\,du = \frac{1}{3}u^3 + C$$

$$= \boxed{\frac{1}{3}(x^2+1)^3 + C}$$

Feb 2-11:05 AM

$$\int (2x+1)(x^2+x+4)^{10}\,dx$$

Let $u = x^2 + x + 4$

$du = (2x+1)\,dx$

$$= \int u^{10}\,du = \frac{u^{11}}{11} + C = \boxed{\frac{1}{11}(x^2+x+4)^{11} + C}$$

$$\int x^3 \cos(x^4+1)\,dx$$

Let $u = x^4 + 1$

$du = 4x^3\,dx$

$\frac{du}{4} = x^3\,dx$

$$= \int \cos u \cdot \tfrac{1}{4}\,du$$

$$= \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C = \boxed{\frac{1}{4}\sin(x^4+1) + C}$$

Feb 2-11:11 AM

Evaluate $\int x \sin x^2\,dx$

$u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$= \int \sin u \cdot \frac{du}{2}$$

$$= \frac{1}{2}\int \sin u\,du = \frac{1}{2}\cdot -\cos u + C = \boxed{\frac{-1}{2}\cos x^2 + C}$$

Evaluate $\int \cos x \sin(\sin x)\,dx$

$u = \sin x$

$du = \cos x\,dx$

$$= \int \sin u\,du = -\cos u + C$$

$$= \boxed{-\cos(\sin x) + C}$$

Verify

$$\frac{d}{dx}\left[-\cos(\sin x) + C\right] = --\sin(\sin x)\cdot\cos x + 0$$

$$= \cos x \sin(\sin x)$$

Feb 2-11:19 AM

$$\int (2x+1)\sqrt[3]{x^2+x}\,dx = \int \sqrt[3]{u}\,du$$

$u = x^2 + x$

$du = (2x+1)dx$

$$= \int u^{1/3}\,du$$

$$= \frac{u^{4/3}}{4/3} + C$$

$$= \frac{3}{4}u\sqrt[3]{u} + C$$

$$= \boxed{\frac{3}{4}(x^2+x)\sqrt[3]{x^2+x} + C}$$

$$2\int \frac{\sin\sqrt{x}}{2\sqrt{x}}\,dx$$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$$= 2\int \sin u\,du = 2\cdot -\cos u + C = \boxed{-2\cos\sqrt{x} + C}$$

Feb 2-11:28 AM

Substitution Method:

$$\int f(g(x))\cdot g'(x)\,dx = \int f(u)\,du$$

where $u = g(x)$.

$$\int \sec^2(\sin x)\cdot\cos x\,dx =$$

$u = \sin x$

$du = \cos x\,dx$

$$\int \sec^2 u\,du = \tan u + C = \boxed{\tan(\sin x) + C}$$

verify:

$$\frac{d}{dx}\left[\tan(\sin x) + C\right] = \sec^2(\sin x)\cdot\cos x$$

Feb 2-11:35 AM

Evaluate $\displaystyle\int_0^{\sqrt{\pi}} x\cos x^2\,dx$

$u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$x = 0 \quad u = 0^2 = 0$

$x = \sqrt{\pi} \quad u = (\sqrt{\pi})^2 = \pi$

$$\int_0^{\pi} \cos u \,\frac{du}{2}$$

$$= \frac{1}{2}\cdot \sin u \Big|_0^{\pi} = \frac{1}{2}\left[\cancel{\sin \pi}^{\,0} - \cancel{\sin 0}^{\,0}\right] = \boxed{0}$$

Feb 2-11:42 AM

Evaluate $\displaystyle\int_{1/6}^{1/2} \csc \pi x \cot \pi x \,dx$

Hint: $u = \pi x$

$du = \pi\,dx$

$\frac{du}{\pi} = dx$

$x = 1/6 \to u = \frac{\pi}{6}$

$x = \frac{1}{2} \to u = \frac{\pi}{2}$

$$\int_{\pi/6}^{\pi/2} \csc u \cot u \,\frac{du}{\pi}$$

$\csc A = \frac{1}{\sin A}$

$\csc \frac{\pi}{2} = \frac{1}{\sin \pi/2} = \frac{1}{1} = 1$

$\csc \frac{\pi}{6} = \frac{1}{\sin \pi/6} = \frac{1}{1/2} = 2$

$$= \frac{1}{\pi}\cdot -\csc u \Big|_{\pi/6}^{\pi/2}$$

$$= \frac{-1}{\pi}\left[\cancel{\csc \tfrac{\pi}{2}}^{\,1} - \cancel{\csc \tfrac{\pi}{6}}^{\,2}\right]$$

$$= \frac{-1}{\pi}\left[1 - 2\right] = \boxed{\frac{1}{\pi}}$$

Feb 2-11:47 AM

Evaluate $\int_0^1 \sqrt[3]{7x+1}\,dx$

$u = 7x+1$

$du = 7\,dx$

$\frac{du}{7} = dx$

$x=0 \to u=1$

$x=1 \to u=8$

$$\int_1^8 \sqrt[3]{u}\,\frac{du}{7}$$

$$=\int_1^8 u^{1/3}\,\frac{du}{7}$$

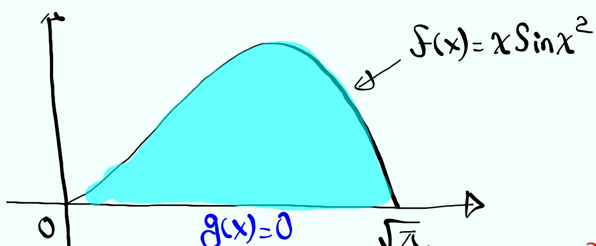
$$=\frac{1}{7}\cdot\frac{u^{4/3}}{4/3}\Bigg|_1^8 = \frac{3}{28}\,u\sqrt[3]{u}\Bigg|_1^8$$

$$=\frac{3}{28}\left[8\sqrt[3]{8} - 1\sqrt[3]{1}\right]$$

$$=\frac{3}{28}\left[16-1\right] = \boxed{\frac{45}{28}}$$

Feb 2-11:54 AM

Find the shaded area below



$u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$x=0 \quad u=0^2=0$

$x=\sqrt{\pi} \quad u=(\sqrt{\pi})^2=\pi$

$$A = \int_0^{\sqrt{\pi}} x\sin x^2\,dx$$

$$=\int_0^{\pi} \sin u\,\frac{du}{2}$$

$$=\frac{1}{2}\left[-\cos u\Big|_0^{\pi}\right]$$

$$=\frac{-1}{2}\left[\cos\pi - \cos 0\right] = \frac{-1}{2}\left[-2\right] = \boxed{1}$$

Feb 2-12:01 PM